Co-funded by the Erasmus+ Programme of the European Union

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Malthus’s Law: In Biology, Malthus’s Law is known as a model for population growth, which belongs to the group of logistic equations. Let x(t) denote the number of individuals in a given population (for example, the population of bacteria) at the time moment t, which are living in one environment. The population can be a human population or some kind of animal population. After a time interval of Δt, the population size is changed to $x(t+\Delta t)$. The change in the number of individuals in the population for the time interval Δt is given with the difference $\Delta x = x(t+\Delta t) - x(t)$. We can be approximately assumed that the number of born individuals during time Δt is equal to the product $nx(t)\Delta t$. The number of dead individuals during time Δt is equal to the product $mx(t)\Delta t$. The equation will be
$$\Delta x = nx(t)\Delta t - mx(t)\Delta t$$
$$\Delta x = (n-m)x(t)\Delta t$$
where n is the coefficient of birth rate and m is the coefficient of a death rate. By the change k=n-m, the equation has a form
$$\Delta x = k\, x(t)\Delta t$$
By dividing the equation from two sides with Δt, the equation has the following form
$$\frac{\Delta x}{\Delta t} = k\, x(t)$$
By using the definition for the existence of the first derivative
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	$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$ the ODE $$\frac{dx}{dt} = k\,x(t)$$ is obtained, where k is constant. In the time t=0, the initial number of individuals in the population is $x(0)=x_0$. In the end, the ODE has a form $$\frac{dx}{dt} = k\,x(t),\ x(0)=x_0$$ and a solution $$x(t) = x_0 e^{kt}.$$ The parameter k gives us the difference between the coefficients of the birth rate and the death rate. If k>0 then the population has exponential growth during time t. If k<0 then the population declines during time t. But, population growth or decline is a complex process and it depends on other factors like temperature, diseases, immigration and etc.	☐Presentation ☐Project ☑Published work
Previous knowledge assumed:	• Knowledge of Trigonometry • Knowledge of Inverse Trigonometric Functions • Knowledge of Logarithmic Function • Knowledge of Differentiation • Knowledge of Integration	
Introduction / Theoretical basics	• Short repetition Discussion questions between the teacher and the students. For each area, questions for the students: ❖ Trigonometry ➢ How are trigonometric functions defined from an acute angle in a right-angled triangle? ➢ Write the values of the trigonometric functions from the standard angles 0^0, 30^0, 45^0, 60^0, 90^0 ? ➢ Write the basic trigonometric	

	identities from the same angle? ➢ Transform $\sin(90^0-\alpha)$=?, $\cos(90^0-\alpha)$=?, $tg(90^0-\alpha)$=?, $ctg(90^0-\alpha)$=? ! ➢ Determine the domain and the set of values of the trigonometric functions ! ➢ How are the functions defined sec x and cosec x ? ❖ Inverse Trigonometric Functions ➢ Write the inverse trigonometric functions ! ➢ Determine the domain and the set of values of the inverse trigonometric functions! ➢ Sketch their graphs by comparing them with the corresponding graphs of the trigonometric functions! ❖ Logarithmic Function ➢ Define a logarithmic function ! ➢ Which logarithm is the natural logarithm ? ➢ Which logarithm is the common logarithm ? ➢ Write the logarithm rules ! ❖ Differentiation ➢ When the function y = f (x) is differentiable at the point a ? ➢ Define the first derivative of the function y=f(x) ! ➢ Write the rules for differentiation ! ➢ Write some derivatives from the derivative table ! ➢ How will we write the following forms y', y'', y''' ? ➢ How are the texts of Rolle's theorem and mean value theorem ! ❖ Integration	

	➢ Which operation is inverse to differentiation ? ➢ Types of integrals ! Which integral is indefinite integral ? Which integral is definite integral ? ➢ In the record $\int f(x)dx$, what is f(x), and what is x ? ➢ If $\frac{d}{dx}F(x) = f(x)$ then$\int f(x)dx =$? ➢ Write some integrals from the integration table ! ➢ Which integration methods do you remember ? ➢ Compare differentiation and integration. What are the obvious conclusions ? For first time in the work of Newton and Leibniz in 1671, the differential equations are appeared. They have made progress since then. As sources for obtaining differential equations there are basic physical laws, natural and technical phenomena (phenomena). We can say that all nature can be expressed (modeled) mathematically with differential equations. The differential equations are appeared in tasks of physics, chemistry, biology, engineering, economics, ecology, medicine, and ect. • Practical problems The importance of the differential equations as mathematical models of the natural laws, we will explain via a practical problem in Biology. The same differential equation is related to a similar problem in Chemistry. For the third problem in Physics, students are asked to research at home in the form of homework. ➢ Malthus Law as a practical problem in Biology, we describe in this part of the class (see the part "Description of the practical problem");	

	➢ We explain that in Chemistry there is a law of radioactivity given by the differential equation $$\frac{dx}{dt} = -\,k\,x(t)$$ where k>0 is the coefficient of radioactivity with the solution $$x(t) = Ce^{-kt}$$ Why do we have a sign in the equation "-"? Because over time the mass of matter that decomposes through a chemical reaction decreases. So the radioactive decay takes place according to the "exponential law". ➢ Newton's second law in the dynamics of the material point.	
Action	The main part of the lesson takes place by presenting a definition of a differential equation and with basic concepts of ODE. During the presentations, there is discussion via questions between the teacher and the students. The theory will be supported by examples, where students will have the keyword. • Definition **Definition.** The equation which include independent variables $x_1, x_2, \dots, x_m$, unknown function $y = y(x_1, x_2, \dots, x_m)$ and its partial derivatives with respect to the independent variables with a form $$F(x_1, x_2, \dots, x_m, y, \frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_m}, \frac{\partial^2 y}{\partial {x_1}^2}, \dots, \frac{\partial^2 y}{\partial {x_m}^2}, \frac{\partial^2 y}{\partial x_1 \partial x_2}, \dots, \frac{\partial^2 y}{\partial x_{m-1} \partial x_m}, \dots, \frac{\partial^n y}{\partial {x_1}^n}, \dots, \frac{\partial^n y}{\partial {x_m}^n}) = 0$$ is called a *differential equation*. When m≥2, the unknown function $y = y(x_1, x_2, \dots, x_m)$ is a function of two or more independent variables then this differential equation is called *partial differential equation* (or	

briefly a PDE). When m=1, the unknown function $y = y(x)$ is a function of one independent variables then this differential equation is called *ordinary differential equation* (or briefly an ODE).

In the further course of the class, we will debate only for an ODE. For an ODE, the following more precise definition is given:

Definition. Let $y = y(x)$ is an unknown function of one independent variables x and y', y'', … , $y^{(n)}$ its derivatives by x. The equation

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$$

or

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0 \quad (1)$$

is called *ordinary differential equation* (or briefly an ODE).

Example 1. a) For the unknown function $z = z(x, y)$, PDE are

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = 2x - y$$

b) For the unknown function $y = y(x)$, ODE are:

$$y''' = \frac{lnx}{x^2}$$

$$y'' + 4y = 0$$

$$y' = e^x + 1$$

Remark: In the following text, instead of an ordinary differential equation, we will use only a differential equation or only an equation.

- The order of a differential equation

The largest derivative of the function $y = y(x)$, which is included in the equation denotes its *order*. The differential equation (1) is of *order n*.

In the differential equation of order n does not have to be included the independent variable x, the function y, and/or some of the derivatives of the function y, except the derivative from order n.

Example 2. In Example 1, б) which order do the differential equations have? Cleary, the equation $y''' = \frac{lnx}{x^2}$ is a differential equation of order three. The equation $y'' + 4y = 0$ is a differential equation of order two and $y' = e^x + 1$ is a differential equation of order one.

- Solution of differential equation

For introducing the term a solution of a differential equation, we will consider the following Example 3:

Example 3. We will consider one algebraic and one differential equation and we will compare their solutions.

a) The algebraic equation

$$x^2 - x - 2 = 0$$

is a quadratic equation with solutions $x_1 = -1$ and $x_2 = 2$. The set of the solutions of this algebraic equation is the set of numbers {-1,2}. The check is very simple:

$$x^2 - x - 2 = (-1)^2 - (-1) - 2 = 0$$
$$x^2 - x - 2 = 2^2 - 2 - 2 = 0$$

b) The differential equation

$$y'' + y' = 2y$$

or with the following forms

$$y''(x) + y'(x) = 2y(x)$$
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2y$$

Has solutions $y_1(x) = e^x$ и $y_2(x) = e^{-2x}$. The set of the solutions of this differential equation is the set of functions $\{y_1(x), y_2(x)\}$. Their check is more complex than the check of the solutions for the algebraic equations. Firstly we require the first and the second derivative of the functions which are solutions for differential equations.

They are

$$y_1'(x) = y_1''(x) = e^x$$
$$y_2'(x) = -2e^{-2x}, y_2''(x) = 4e^{-2x}$$
In the differential equation, the first solution $y_1(x) = e^x$ with its first and second derivatives will be included
$$y_1''(x) + y_1'(x) = 2y_1(x)$$
$$e^x + e^x = 2e^x, \forall x \in (-\infty, +\infty)$$
The function $y_1(x) = e^x$ identically satisfies the differential equation. We conclude that the function $y_1(x) = e^x$ is a solution for the differential equation. On this class, the check for the second solution $y_2(x) = e^{-2x}$ is left of the students.

Question for the students:

- What is the difference between the solutions of an algebraic equation and the solutions of the differential equation given in Example 3?

For the differential equation in Example 3, we conclude:
A solution of the differential equation
$$y'' + y' = 2y$$
is the set of the functions $\{y_1(x), y_2(x)\}$.
Solving or integrating of the differential equation $y'' + y' = 2y$, means to find the set of functions that satisfy the equation.

Remark: The method for obtaining of the solutions of this differential equation, we will leave for later lessons.

Because the solutions are functions of one independent variable, we wonder for their geometric interpretation.

The question for the students:

- What is the geometric interpretation of its solutions $y_1(x) = e^x$ и $y_2(x) = e^{-2x}$?

It is clear that the geometric interpretation is the curves of the functions $y_1(x) = e^x$ и $y_2(x) =$

e^{-2x} in the coordinate system Oxy. As the solutions of the differential equation $y'' + y' = 2y$ are called integral curves for a given differential equation.

Generalization:

For each function $y = y(x)$ defined on some interval (a,b) which identically satisfied the equation (1) is called a *solution* of equation (1) i.e.

$$F\left(x, y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}, \dots, \frac{d^ny(x)}{dx^n}\right) = 0,$$
$$\forall x \in (a, b)$$

The curve $y = y(x)$ which presented a geometric solution of the equation (1) is called *an integral curve*. Solving or integrating of the differential equation (1), means to find all functions $y = y(x)$ that identically satisfy the equation (1).

Next Example 4 is for the students.

Example 4. Are the functions $y = C_1x^2 + C_2(x-1)$ and $y = x^2$ solutions for the differential equation

$$(x^2 - 2x)y'' - 2(x-1)y' + 2y = 0$$

where C_1 and C_2 are arbitrary constants.

The questions for the students:

- What did you conclude after the check?
- You have two solutions for the equation. Can the second solution obtain from the first solution for the concrete values of the arbitrary constants? If it can do than you find these values!

For the equation (1) there is a solution which contains n arbitrary essential constants $\mathrm{C}_1, \mathrm{C}_2, \dots, \mathrm{C}_n$ in an explicit form

$$y = \varphi(x, \mathrm{C}_1, \mathrm{C}_2, \dots, \mathrm{C}_n) \quad (2)$$

	Or an implicit form $\psi(x, C_1, C_2, \ldots, C_n) = 0$ (3) The equations (2) and (3) are called *general solution* for the equation (1) which depends on n arbitrary constants $C_1, C_2, \ldots, C_n$. The general solution of the equation (1) is an integral curve in the plane that depends on n arbitrary essential constants essentially $C_1, C_2, \ldots, C_n$ so-called *integration constants*. **Remark:** The word **"**essential constants" or "independents" constants, mean they can not present with less number of constants from given constants. **Example 5.** In the function $= sinx + A + B$, the constants A and B are not essential constants, because with the change C=A+B is obtained only one constant C. In the function $y = sinx + C$, the constant C is an essential constant. In equations (2) and (3), if one or more arbitrary constants are replaced with concrete values then the equations (2) and (3) will be called *particular equations* for the equation (1). So, the particular integral can be obtained from the general solution by putting some concrete values for the integration constants. The values of integration constants from the general solution can be obtained when for the function y and its derivatives y', y'',…,$y^{(n-1)}$ are replaced with the concrete values $y_0, y_0', \ldots, {y_0}^{(n-1)}$ for $x = x_0$.The values $y_0, y_0', \ldots, {y_0}^{(n-1)}$ for $x = x_0$ are called the initial values or the initial conditions for the equation (1). Question for the students: ➢ In Example 4, which solution is general solution, but which is particular for the given differential equation ? An answer: $y = C_1x^2 + C_2(x-1)$ is a general solution and $y = x^2$ is a particular solution for the differential equation $(x^2 - 2x)y'' - 2(x-1)y' + 2y = 0$	

	The particular equation $y = x^2$ is obtained from the general solution $y = C_1x^2 + C_2(x-1)$ for $C_1 = 1$ and $C_2 = 0$. These values of the integration constants are obtained from the general solution of the differential equation with the initial values y=1, y'=2 for x=1 or short written y(1)=1, y'(1)=2. • Formation of differential equation Many natural sciences and applied mathematics tasks give differential equations. The most elementary is to obtain a differential equation from the equation of an unknown function and by requesting it's a derivative without taking into account any specific interpretation of it. **Example 6.** By elimination of the constant b in the unknown function $y = bx^3$ via its first derivative $y' = 3bx^2$ i.e. $b = \frac{y'}{3x^2}$ in the function $y = \frac{y'}{3}x$ is obtained the differential equation $y' = \frac{3y}{x}$. Question for the students: ➢ What does a function $y = bx^3$ present for the differential equation $y' = \frac{3y}{x}$? We will compose a differential equation for a given family of curves. Let the family of curves, which depends from n arbitrary constants $C_1, C_2, \ldots, C_n$ with form $f(x, C_1, C_2, \ldots, C_n) = 0 \quad (4)$ is given. By requesting of all n derivatives of the function (4) per x, we obtained the following n equations: $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' = 0$ $\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\,\partial y}y' + \frac{\partial^2 f}{\partial y^2}y'^2 + \frac{\partial f}{\partial y}y'' = 0$ $\cdots\cdots\cdots\cdots\cdots\cdots$ $\frac{\partial^n f}{\partial x^n} + \cdots + \frac{\partial f}{\partial y}y^{(n)} = 0$	

The n equations with the equation (4) form n+1 equations. By elimination of the n constants, the equation

$$\varphi(x, y, y', y'', \dots, y^{(n)}) = 0 \quad (5)$$

is obtained.

From the way we came to this equation, it is clear that every function with a form (4) satisfies the equation (5) for each x and for every values of the arbitrary constants $C_1, C_2, \dots, C_n$ i.e. the equation (4) is a *solution* for the equation (5). If the constants $C_1, C_2, \dots, C_n$ c are arbitrary then (4) is a *general solution* of equation (5). If the constants have concrete values then we are talking about a *particular solution*. The equation (4) represents a family of curves in the plane called *integral curves* for the equation (5), and the equation (5) is a differential equation of that observed family of curves. From the differential equation (5), we can be seen that its order is equal to the number of the arbitrary constants which contained in its general solution.

Next: we will give an example for formation of a differential equation in an applied mathematics.

Example 7. Let us find the differential equation of the family of lines in the plane given by the equation $y = ax + b$, where a and b are parameters. We request a first and a second derivative for the function $y = ax + b$ per x, $y' = a, y'' = 0$. The given equation which presents a family of lines in the plane with a form $y = ax + b$ is $y'' = 0$.

Question for the students:

- What does the function $y = ax + b$ present for the differential equation $y'' = 0$?

Finally, we can give the following conclusion for solving problems from applied mathematics, natural sciences and technology.

For the question:

	➢ What it means to solve a task from the applied mathematics, natural sciences and technology ? The answer is: The solution of the task is in following steps. 1. Based on a detailed overview of the conditions of the task (if it need can be drawn a drawing) a *differential equation is composed*. This is usually the easiest part of the task; 2. By help of mathematical method, the *differential equation is solved* (obtaining a general and a particular solution according to the conditions of the task). This is usually the hardest part of the task; 3. The obtained solution of the differential equation *is checked* and *analyzed*. The analysis gives conclusions for *predicting future processes* and *reconstruction of past processes*.	
Materials / equipment / digital tools / software	*The materials for learning* are given as a part of references of the end from this topic plan; *Equipment*: classroom, green board, chalk in different colours; *Digital tools*: laptop, projector, smart board; *Software*: Mathematica.	
Consolidation	• Use of materials, equipment, digital tools, software by teachers and students; • The teacher's discussion with the students through appropriate questions; • Independent solving of simple tasks by the students under the supervision of the teacher; • Given of examples by the teacher for introducing a new concept in a cooperation and a discussion with the students; • Assignment of homework by the teacher with a time limit until the next class.	
Reflections and next steps		
Activities that worked	**Parts to be revisited**	

After the class, the teacher according to his personal perceptions regarding the success of the class fills in this part.	Through the success of the homework done by the students, questions and discussion at the beginning of the next class, the teacher comes to the conclusion which parts of this class should be revised.
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